Generalized Invariants of a 4^{th} order tensor: Building blocks for new biomarkers in dMRI

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Abstract. This paper presents a general and complete (up to degree 4) set of invariants of 3D 4^{th} order tensors with respect to $\mathcal{SO}_3$. The invariants to $\mathcal{SO}_3$ for the 2^{nd} order diffusion tensor are well known and play a crucial role in deriving important biomarkers for DTI, e.g. MD, FA, RA, etc. But DTI is limited in regions with fiber heterogeneity and DTI biomarkers severely lack specificity. 4^{th} order tensors are both a natural extension to DTI and also form an alternate basis to spherical harmonics for spherical functions. This paper presents a systematic method for computing the $\mathcal{SO}_3$ invariants of 3D 4^{th} order tensors, derives relationships between the new (generalized) invariants and existing invariants and shows results on synthetic and real data. It also presents, hitherto unknown, new invariants for 4^{th}order tensors. Analogously to DTI, these new invariants can perhaps form building blocks for new biomarkers.

Keywords: Invariants, 4^{th} order tensors, homogeneous forms/polynomials, $\mathcal{SO}_3$, basic invariants, principal invariants, biomarkers.

1 Introduction

The invariants of a 2^{nd} order diffusion tensor, to $\mathcal{SO}_3$ the rotation group, play a fundamental role in diffusion tensor imaging (DTI). They form the building blocks for a wide range of popularly used biomarkers such as mean diffusivity (MD), fractional anisotropy (FA), relative anisotropy (RA), etc. [1]. The importance of the diffusion tensor's invariants is highlighted by the importance of biomarkers in diffusion MRI (dMRI). Biomarkers are crucial in analyzing changes in white matter related to development, degeneration and disease.

However, DTI is limited in regions with heterogeneous fiber configurations. Furthermore, DTI based biomarkers severely lack specificity – not only due to the coarse resolution of dMRI with respect to the actual axons, but also due to the over-simplification of the DTI model. Therefore, the MD or the FA are affected similarly by a variety of disparate types of physical changes or acquisition settings (partial voluming). Hence, there is a strong need for conceiving of new scalar measures for characterizing the integrity of white matter – especially from higher order models which describe the microstructure with greater accuracy.

Higher order tensors and in particular 4^{th} order tensors are a natural generalization of the 2^{nd} order diffusion tensor. Initially proposed in generalized DTI (GDTI) [2] for describing complex apparent diffusion coefficient (ADC) profiles, they also appear as a bijective alternate basis to the spherical harmonic basis for describing generic functions on the sphere [2]. 4^{th} order tensors, therefore, can be used for describing a number of well-known spherical functions from higher order models in dMRI. Furthermore, 4^{th} order tensors have been extensively studied and important mathematical frameworks have been developed for estimating ADCs with the positivity constraint [3–6]

In this paper, we present the general and a complete (up to degree 4) set of invariants of 3D 4^{th} order tensors with respect to $\mathcal{SO}_3$. There exists no simple approach for either computing or for interpreting these invariants as for 2^{nd} order tensors. However, analogously to DTI, we believe that these invariants would form the building blocks for new biomarkers. We call these the "general" invariants to distinguish them from the integrity basis of 4^{th} order tensors that only compute certain types of invariants – the basic ($S4_i$) and the principal ($J4_i$) [7, 8]. To our knowledge, this approach has never been presented before.

There exist two known methods for computing the invariants of 3D 4^{th} order tensors to $\mathcal{SO}_3$. In [9] the authors integrate a function of the tensor over the sphere to compute a scalar measure (generalized trace) invariant under 3D rotations. This provides a way for computing the invariants. However, there is no systematic way for choosing which functions to integrate, therefore, there is no systematic way of generating a linearly independent set of invariants using this method. In [7, 8] the authors compute the integrity basis – the 6 basic and the algebraically dependent 6 principal invariants from the spectral decomposition of the 6×6 matrix representation of the 4^{th} order tensor. However, these are designed to be invariant to $\mathcal{SO}_6$. Therefore, this approach is limited since symmetric 3D 4^{th} order tensors have 15 independent coefficients and any 3D rotation can be described by 3 parameters, implying that 3D 4^{th} order tensors should have $12 = 15 - 3$ algebraically independent invariants to $\mathcal{SO}_3$.

In this paper, we work out a systematic approach for computing the polynomial invariants which are complete up to a given degree. Our approach is based on the polynomial interpretation of tensors. It is well known that tensors can be rewritten as homogeneous polynomials or "forms" by a rearrangement of the indices. For example a 3D 4^{th} order tensor is simply a ternary quartic form – a homogeneous degree-4 polynomial in 3 variables. After presenting the theory and deriving their formulae, we conduct experiments on synthetic datasets in an attempt to understand these invariants and finally present results on an in vivo human cerebral dataset.

2 Materials and Methods

This section describes a systematic method to generate the polynomial invariants of a form under the group $\mathcal{SO}_3$ of the rotations of $\mathcal{R}^3$. The method will be illustrated with the simple example of 2-forms (quadratic forms). Then, the results of applying the method to 4-forms (quartic forms) will be described.

2.1 Invariants of a form under group transformations

Let $\mathcal{G}$ denote a transformation group operating on $\mathcal{R}^3$. We further assume that $\mathcal{G}$ is a Lie group, i.e. a group that depends continuously on set of n parameters. Let $\mathcal{H}$ be a family of objects of $\mathcal{R}^3$ depending on p parameters. Since $\mathcal{G}$ operates on $\mathcal{R}^3$, it induces transforms over the family $\mathcal{H}$. Invariants of $\mathcal{H}$ under the group $\mathcal{G}$ are expressions involving the p parameters of $\mathcal{H}$ that remain unchanged under the transformations induced by the group $\mathcal{G}$. Polynomial invariants correspond to those expressions which are polynomials in the p parameters of $\mathcal{H}$. Invariant theory states that all polynomial invariants can be expressed as polynomial expressions of a finite set of $p - n$ basic polynomial invariants.

As an example, 4^{th} order forms over $\mathcal{R}^3$, or ternary quartics, depend on $p = 15$ parameters. Thus, there are $12 = 15 - 3$ basic polynomial invariants of 4^{th} order forms under the rotation group of $\mathcal{R}^3$ (n=3).

2.2 Generators of $\mathcal{SO}_3$

The method proposed in the next section is quite general but relies on rational parameterization of the group $\mathcal{G}$. Such a parameterization is thus described hereafter for $\mathcal{SO}_3$, which is the group of interest in the context of this work. Euler angles is a way to parameterize rotations. Any rotation $\mathbf{R}$ of $\mathcal{SO}_3$ can be written as the composition of three elementary rotations with angles α, β, γ:

$$\mathbf{R} = \mathbf{R}_Z(\alpha)\mathbf{R}_Y(\beta)\mathbf{R}_Z(\gamma) \ , \tag{1}$$

where $\mathbf{R}_Y(\theta)$ and $\mathbf{R}_Z(\theta)$ denote respectively rotations around the Y and Z axis with the angle θ. Introducing $t = tan\frac{\theta}{2}$, $\mathbf{R}_Y(\theta)$ and $\mathbf{R}_Z(\theta)$ can be written as:

$$\mathbf{R}_Y(\theta) = \begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{bmatrix} \ , \quad \mathbf{R}_Z(\theta) = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ with } c = \frac{1-t^2}{1+t^2} \text{ and } s = \frac{2t}{1+t^2} \ .$$

Because of Eq. (1), a polynomial expression separately invariant under both the two sub-families $\mathbf{R}_Y(\theta)$ and $\mathbf{R}_Z(\theta)$ (for any values of θ) is also invariant under the whole group $\mathcal{SO}_3$. The next section describes how to find such polynomial expressions. It will be applied sequentially with the subgroups $\mathbf{R}_Y(\theta)$ and $\mathbf{R}_Z(\theta)$ to generate polynomial expressions invariant under $\mathcal{SO}_3$.

2.3 Systematic generation of polynomial invariants

Let us denote by $\mathbf{S}(t)$ the group under which invariants are sought. t is the parameter which allows the rational representation of the Lie group. Let $\mathbf{m} = [x,\ y,\ z]$ be a vector of the 3D space. Let $f_q(\mathbf{m}) = \sum_{i+j+k=q} \mu_{ijk} c_{ijk} x^i y^j z^k$ denote a homogeneous form of degree q with coefficients c_{ijk} (there are p such coefficients). μ_{ijk} are constants eventually used to represent weights of the monomial $x^i y^j z^k$.

For example, a 2-form is defined by $f_2(\mathbf{m}) = c_{200}x^2+2c_{110}xy+2c_{101}xz+c_{020}y^2+2c_{011}yz + c_{002}z^2$ ($p_2 = 6$).

For every t, $\mathbf{S}(t)$ defines a change of coordinates $\mathbf{m}' = \mathbf{S}(t)\mathbf{m}$. Applying this change to f_q yields a new form of degree q $f'_q(\mathbf{m}') = \sum_{i+j+k=q} \mu_{ijk} c'_{ijk} x^i y^j z^k$. c'_{ijk} are rational expressions in t and linear expressions in the coefficients c_{ijk}.

Let $I_d(\{c_{ijk}\})$ be a polynomial expression of degree d with coefficients $m_r, r = 1..R$. Without loss of generality, I_d can be restricted to be a homogeneous polynomial. I_d is invariant iff $I_d(\{c_{ijk}\}) - I_d(\{c'_{ijk}\})$=0. The numerator of this expression is a polynomial in t and c_{ijk} which must be identically 0. Consequently, each of its coefficients is null yielding a set of linear constraints on the coefficients m_r. This system potentially has multiple solutions providing a generative family of the polynomial invariants of degree d. The full algorithm is described in 1.

Algorithm 1 Generation of a basis of the set of homogeneous polynomial invariants of degree d of a form f under $\mathcal{SO}_3$.

Require: A polynomial expression $f(\mathbf{m})$ with coefficients c_{ijk}.
  Start with a generic homogeneous polynomial $I_d(\{c_{ijk}\})$ of degree d in the variables c_{ijk} with coefficients m_{ijk}.
  Initialize the parameter list $P = \{m_{ijk}\}$.
  for all $\mathbf{S}(t) \in \{\mathbf{R}_Z(t), \mathbf{R}_Y(t)\}$ **do**
    Compute coefficients c'_{ijk} of $f(\mathbf{S}(t)\mathbf{m})$ induced by the change of coordinates $\mathbf{S}(t)$.
    Compute the numerator of $D(t, P) = I_d(\{c_{ijk}\}) - I_d(\{c'_{ijk}\})$.
    Extract all the coefficients of the polynomial $D(t, P) \rightarrow L$.
    ▷ *L is a linear system in the coefficients of P.*
    Solve L and substitute the solution in $I_d(\{c_{ijk}\})$.
    ▷ *L is not a square system, its solution contains some of the unknowns m_{ijk}.*
    Update P as the list of parameters in the solution of L.
  end for
  Extract the coefficients of I_d with respect to $P \rightarrow Invs$.
  ▷ *$Invs$ is a basis of the set of the sought polynomial invariants.*
  return $Invs$

Considering the 2-form f_2, an invariant of degree 1 can be written as $I_d(\{c_{ijk}\}) = \sum_{l \in Ind} m_l c_l$, with $Ind = \{200, 020, 002, 110, 101, 011\}$ and $P = \{m_l, l \in Ind\}$. Using the family of rotations $\mathbf{R}_Z(t)$, the numerator of $I_d(\{c_{ijk}\}) - I_d(\{c'_{ijk}\}) = 0$ (divided by $2t$) can be written as:

$$2(m_{200} - m_{020}) \left((c_{020} - c_{200})\, t - c_{110} \left(1 + t^2\right) \right) + m_{110} \left(1 - t^2\right) (c_{020} - c_{200}) - \left(1 + t^2\right) (c_{011} (m_{011} t - m_{101}) + c_{101} (m_{101} t + m_{011})) - 4 m_{110} c_{110} t = 0 \ ,$$

which yields $m_{200} = m_{020}, m_{110} = m_{011} = m_{011} = 0$. This means that $I_d(\{c_{ijk}\}) = m_{200} (c_{200} + c_{020}) + m_{002} c_{002}$ with a new set of parameters $P = \{m_{200}, m_{002}\}$. Applying the same procedure to this new polynomial with $\mathbf{R}_Y(\theta)$ yields the familiar trace invariant $c_{200} + c_{020} + c_{002}$. In this case, the last linear system has only one solution (up to an arbitrary scale factor) $m_{200} = m_{002}$.

2.4 Linear independence and simplification of the invariants

Using algorithm 1, we get I_d a basis of all homogeneous polynomial invariants of degree d. Yet nothing ensures that those invariants are independent of those of lower degrees. This section explains how to construct a linearly independent set of invariants up to degree d. Algorithm 2 is recursive: assuming that a linearly independent set J_{d-1} of invariants up to degree $d-1$ is available, it computes J_d given I_d. In this paper, J_1 is always non empty set and contains one single polynomial (so it is independent). Algorithm 2 makes use of a function $matrix(K_d)$, where K_d is a set of homogeneous polynomials of degree d. This function returns a matrix whose entries are the coefficients of the polynomials. Each row of the matrix corresponds to a polynomial of K_d. Each column corresponds to a monomial of degree d (ordered arbitrarily). If the rank of this matrix is the same as the number of polynomials in K_d, then the system K_d is linearly independent.

Algorithm 2 Generation of a basis of the set of polynomial invariants up to degree d.

```
Initialize J_d = J_{d-1}.
Compute a basis K_d of the homogeneous polynomial of degree d generated by J_{d-1}.
▷ I_d is assumed to be sorted in ascending order in terms of simplicity, where
▷ simplicity is for example the number of terms of the polynomial.
for all I_d^i ∈ I_d do
    if rank(matrix(K_d ∪ {I_d^i})) = rank(matrix(K_d)) + 1 then
        ▷ I_d^i is linearly independent from the polynomials in K_d, add it.
        K_d = K_d ∪ {I_d^i}.
        J_d = J_d ∪ {I_d^i}.
    end if
end for
return J_d
```

Note that $M_d = matrix(J_d)$ can also be used to minimize the number of terms in the polynomials in J_d. Indeed, each column of M_d corresponds to a linear system on the polynomials in J_d which cancels the monomial corresponding to that column. Thus all columns provide a set of linear equations. This set can be explored combinatorially to find the linear combinations of the polynomials in J_d which cancel the most monomials.

2.5 Application to 2^{nd} order forms

Given f_2, we introduce the two vectors $\mathbf{V}_1 = [c_{110}, c_{101}, c_{011}]$ and $\mathbf{V}_2 = [c_{200}, c_{020}, c_{002}]$. In this case, it is known that polynomial invariants can be generated using $6 - 3 = 3$ polynomials $J2_1, J2_2, J2_3$ which correspond to the coefficients of the characteristic polynomial $\mathbf{F}$ of the 2^{nd} order tensor associated to f_2 (in particular $J2_1 = trace(\mathbf{F})$ and $J2_3 = det(\mathbf{F})$). Using algorithm 1 for degrees 1, 2 and 3 gives:

$$I_1 = \left\{I_1^1\right\} \qquad \text{with } I_1^1 = J2_1 = c_{002} + c_{020} + c_{200}$$
$$I_2 = \left\{I_2^1, I_2^2\right\} \qquad \text{with } I_2^1 = J2_2 \text{ and } I_2^2 = 2\|\mathbf{V}_1\|^2 + \|\mathbf{V}_2\|^2$$
$$I_3 = \left\{I_3^1, I_3^2, I_3^3\right\} \qquad \text{with } I_3^1 = J2_1^3, I_3^2 = J2_1\left(I_2^1 - I_2^2\right) \text{ and } I_3^3 = J2_3.$$

Algorithm 2 gives successively $J_1 = \{J2_1\}$, $J_2 = \{J2_1, J2_2\}$ and $J_3 = \{J2_1, J2_2, J2_3\}$. J_3 is exactly the set obtained using $\mathbf{F}$. Algorithm 2 also reveals (as expected) that invariants of higher degrees (4 to 10) can all be expressed using those of J_3.

2.6 Application to 4^{th} order forms

Using the proposed method on 4-forms (15 parameters) yields 12 linearly independent invariants up to degree 4. The coefficients of f_4 can be grouped in 5 vectors $\mathbf{V}_1 = [c_{112}, c_{121}, c_{211}]$, $\mathbf{V}_2 = [c_{220}, c_{202}, c_{022}]$, $\mathbf{V}_3 = [c_{130}, c_{301}, c_{013}]$, $\mathbf{V}_4 = [c_{310}, c_{103}, c_{031}]$ and $\mathbf{V}_5 = [c_{400}, c_{040}, c_{004}]$. As in section 2.3, each coefficient c_{ijk} is attached to the monomial $x^i y^j z^k$ and the multiplicity μ_{ijk} are respectively 12, 6, 4, 4 and 1 for the coefficients appearing in respectively $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}_4$ and $\mathbf{V}_5$. Algorithm 2 finds 12 linearly independent invariants up to degree 4: 1 invariant of degree 1, 2 of degree 2, 4 of degree 3 and 5 of degree 4. These "general" invariants will be named $G4_i, i = 1..12$. The first invariants are:

$$G4_1 = 2S_2 + S_5 \ , \qquad G4_2 = 12C_{11} + 6C_{22} + 4(C_{33} + C_{44}) + C_{55} \ ,$$
$$G4_3 = 4(C_{13} + C_{14} + C_{34} - C_{11}) - 3C_{22} - 2(C_{25} + D_2) - D_5 \ ,$$

where $S_i = \sum_{j=1}^{3} \mathbf{V}_{ij}$, $C_{ij} = \mathbf{V}_i \cdot \mathbf{V}_j$, $D_i = \sum_{j \neq k} \mathbf{V}_{ij}\mathbf{V}_{ik}$. The other invariants (up to degree 4) are too complicated to be written here and will be provided on demand as maple code. It has not yet been possible to apply the method of section 2.3 to invariants of degree 5 and 6. While in this paper, we only prove the linear independence between the $G4_i$, it is easy to check that the first 3 invariants (up to degree 2) are actually algebraically (and not only linearly) independent. One of the invariants of degree 2 $G4_3$ is, therefore, new (not a basic/principal invariant). Note that $G4_3$ could still be a consequence of e.g. the principal invariants $J4_i, i = 1..6$ as a power of $G4_3$ might be written as a combination of $\{J4_i\}$. In any case, $G4_3$ is simpler.

2.7 Relation with previously known invariants

The simplification method described in section 2.4 can also be used to express one invariant family as functions of other invariant families. Since the invariants obtained in the previous section are complete up to degree 4, all previously known (basic & principal) invariants (up to degree 4) can be expressed in terms of the invariants $G4_i, i = 1..12$.

$$J4_1 = -\,G4_1\ , \qquad J4_2 = \frac{1}{2}(G4_1^2 - G4_2)\ , \qquad J4_3 = -\frac{1}{6}G4_1^3 + \frac{1}{2}G4_1G4_2 - \frac{1}{3}G4_5$$
$$J4_4 = \frac{G4_1^4}{24} - \frac{5}{12}G4_1^2G4_2 + \frac{1}{3}G4_1G4_5 + \frac{1}{9}G4_1G4_7 + \frac{G4_2^2}{72} - \frac{1}{3}G4_2G4_3 - \frac{G4_9}{36} - \frac{G4_{11}}{504}$$

Thus, the principal invariants up to degree 4 can all be expressed in terms of the general invariants. This cannot be done with $J4_5$ and $J4_6$. Therefore, in the end, we have 16 invariants which are linearly independent. Non-linear dependency among those is much more complicated and under investigation. Using the same method, trace or basic invariants can be expressed as functions of the principal invariants, i.e. basic invariants up to degree 4 are also consequences of the general invariants:

$$S4_1 = -\,J4_1\ , \qquad S4_2 = J4_1^2 - 2J4_2\ , \qquad S4_3 = -J4_1^3 + 3J4_1J4_2 - 3J4_3$$
$$S4_4 = J4_1^4 - 4J4_1^2J4_2 + 4J4_1J4_3 + 2J4_2^2 - 4J4_4$$
$$S4_5 = -\,J4_1^5 + 5J4_1^3J4_2 - 5J4_1^2J4_3 - 5J4_1J4_2^2 + 5J4_1J4_4 + 5J4_2J4_3 - 5J4_5$$
$$S4_6 = J4_1^6 - 6J4_1^4J4_2 + 6J4_1^3J4_3 + 9J4_1^2J4_2^2 - 6J4_1^2J4_4 - 12J4_1J4_2J4_3 + 6J4_1J4_5 - 2J4_2^3 + 6J4_2J4_4 + 3J4_3^2 - 6J4_6$$

3 Experiments and Results

Although any spherical function from higher order models either in the tensor basis or the spherical harmonic basis can be used as a source 4^{th} order tensor, we adopt the simplest GDTI model with least square approximation [2]. However, computing these invariants isn't limited to GDTI.

On synthetic data we test the numerical stability of the 12 invariants to arbitrary 3D rotations to verify their validity. Synthetic diffusion signals in a voxel are generated using $S(\mathbf{g}_i) = \sum_{k=1}^{N=1,2}(1/N)\exp\left(-b\mathbf{g}_i^T\mathbf{D}_k\mathbf{g}_i\right)$ with $\mathbf{D}_k = \mathbf{R}_k^T diag(\lambda_1, \lambda_2, \lambda_2)\mathbf{R}_k$ where $\mathbf{R}_k$ are 3D rotations and $\{\lambda_1 = 2, \lambda_2 = 0.1\} \times 10^{-3}mm^2/s$. Along the x-axis of the dataset we vary the crossing angle from $0^o - 90^o$ in 12 steps. Along the y-axis we vary the anisotropy of each fiber by varying λ_2 from $0.1 - \lambda_1$ in 11 steps. And along the z-axis we vary the volume by increasing each λ_i tenfold over 10 steps.

We generate a second dataset with identical fiber configurations as the above, but with each voxel oriented randomly (Fig. 1a,b). We first estimate the 4^{th} order tensors then compute the 12 invariants for both datasets $\{G4_i\}$ & $\{G4'_i\}$ and finally compute the ratio $|\overline{G4_i}|/|\overline{G4_i - G4'_i}|$ for each invariant. For reference we also conduct the same experiment with FA from 2^{nd} order tensors and with $\{S4_i\}$ & $\{J4_i\}$. The results are presented in Fig. 1c where taller bars indicate bigger ratios and greater numerical stability to rotation. It is interesting to note that 4^{th} order tensor invariants show far greater stability to rotation than FA, which seems to imply that the estimation of 4^{th} order tensors is more stable to rotated diffusion data than the estimation of 2^{nd} order diffusion tensors.

Finally we also conduct experiments on an in vivo human cerebral dataset that was acquired on a 3T Siemens scanner, with 60 gradient directions and a b-value of $1000s/mm^2$ [10]. The resulting 12 invariants are presented in Fig. 1d.

4 Conclusion

We described a method to systematically generate a linearly independent set of homogeneous polynomial invariants up to degree d under $\mathcal{SO}_3$. This method was used to produce the invariants of 2^{nd} and 4^{th} order forms or tensors (for the 4^{th}order only up to degree 4). The tools also allowed us to express the relations between the new invariants and those already known (basic & principal). Furthermore some of the new invariants were found to be algebraically independent of the known invariants, and many were of lower degree. Experimentally we confirmed the numerical stability of the new invariants to $\mathcal{SO}_3$. Future work will explore the algebraic (i.e. non linear) dependency among the new invariants and will attempt to produce a complete set of 12 invariants with simple expressions.

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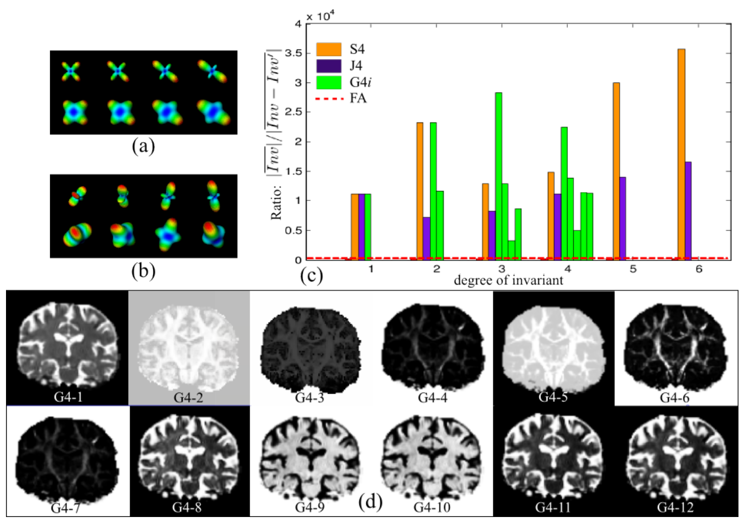

Fig. 1. (a, b) Synthetic data ADC with increasing crossing angles, isotropy and (b) random orientations. (c) Rotation invariance: taller bars (ratios) indicate greater numerical stability, $G4_i$ sorted by degree (sec. 2.6). (d) 12 invariants from human dataset.

References

1. Basser, P.J.: Inferring microstructural features and the physiological state of tissues from diffusion-weighted images. NMR in Biomedicine **8** (1995) 333–344
2. Özarslan, E., Mareci, T.H.: Generalized diffusion tensor imaging and analytical relationships between diffusion tensor imaging and high angular resolution imaging. Magnetic Resonance in Medicine **50** (2003) 955–965
3. Barmpoutis, A., Jian, B., Vemuri, B.C.: Symmetric Positive 4th Order Tensors & their Estimation from Diffusion Weighted MRI. In: Information Processing in Medical Imaging (IPMI 2007). (2007) 308–319
4. Ghosh, A., Descoteaux, M., Deriche, R.: Riemannian framework for estimating symmetric positive definite 4th order diffusion tensors. In Metaxas, D.N., Axel, L., Fichtinger, G., Széke, G., eds.: Medical Image Computing and Computer-Assisted Intervention-MICCAI. Volume 5242 of Lecture Notes in Computer Science., New York, NY, USA, Springer (September 2008) 858–865
5. Barmpoutis, A., Hwang, M.S., Howland, D., Forder, J.R., Vemuri, B.C.: Regularized Positive-Definite Fourth-Order Tensor Field Estimation from DW-MRI. NeuroImage **45**(1 sup.1) (March 2009) 153–162
6. Ghosh, A., Moakher, M., Deriche, R.: Ternary quartic approach for positive 4th order diffusion tensors revisited. In: 2009 IEEE International Symposium on Biomedical Imaging: From Nano to Macro. (June 2009) 618–621
7. Basser, P.J., Pajevic, S.: Spectral decomposition of a 4th-order covariance tensor: Applications to diffusion tensor MRI. Signal Processing **87** (2007) 220–236
8. Fuster, A., van de Sande, J., Astola, L., Poupon, C., Velterop, J., Romeny, B.M.t.H.: Fourth-order Tensor Invariants in High Angular Resolution Diffusion Imaging. In: CDMRI Workshop – MICCAI. Volume 6891 of LNCS., Springer (2011)
9. Özarslan, E., Vemuri, B.C., Mareci, T.H.: Generalized scalar measures for diffusion mri using trace, variance and entropy. Magnetic Resonance in Medicine **53**(4) (2005) 866–876
10. Anwander, A., Tittgemeyer, M., von Cramon, D.Y., Friederici, A.D., Knosche, T.R.: Connectivity-Based Parcellation of Broca's Area. Cerebral Cortex **17**(4) (2007) 816–825